

Chapter 3

Thermodynamics of moist air

Until now we have been dealing with a single-component gas: all molecules are identical. How do things change when two or more different types of molecule are mixed together? If none of the gases can condense, then very little changes, aside from a modified gas constant (see below). However, water can condense at the temperatures typical of Earth's atmosphere. The consequent release of latent heat, and the formation of liquid droplets (clouds) has a huge impact not only on the thermodynamics of the atmosphere, but also its dynamics and radiative transfer. The presence of water makes the atmosphere enormously more complex and interesting, and keeps an army of atmospheric scientists in business.

3.1 Ideal gas law for a mixture of gases: partial pressures

Take two identical boxes of volume V . One contains N_1 molecules of gas 1, the other N_2 molecules of gas 2. If both gases are ideal and are kept at the same temperature, the pressures

inside the boxes will be

$$p_1 = \frac{N_1}{V}kT \quad (3.1)$$

and

$$p_2 = \frac{N_2}{V}kT. \quad (3.2)$$

What happens if we put all the molecules of one box into the other? Ideal gases behave as if each molecule ignored all others. Therefore, the gas already present in the box will not even realise that new gas is being put in; and the gas entering the box will behave as if the box were empty. After the new gas is put in, the pressure is simply the *sum* of the pressures due to each gas:

$$p = p_1 + p_2 = \frac{N_1 + N_2}{V}kT. \quad (3.3)$$

This is known as *Dalton's law of partial pressures*: the total pressure is the sum of the pressures exerted by each of the gases if it occupied the volume alone. The law applies to a mixture with any number of components:

$$p = \frac{\sum_i N_i}{V}kT, \quad (3.4)$$

and defining the mean molecular mass

$$\langle m \rangle = \frac{\sum_i N_i m_i}{\sum_i N_i} = \sum_i f_i m_i, \quad (3.5)$$

where f_i is the number fraction of component i , we can write

$$p = \frac{\sum_i N_i m_i}{V} \frac{k}{\langle m \rangle} T = \rho RT. \quad (3.6)$$

Thus, a mixture of ideal gases behaves just like a single-component ideal gas with gas constant $R = k/\langle m \rangle$.

As we noted in Section 1.1, the number density of atmospheric components other than water vapour is essentially constant in space and time. It is thus useful to define a gas constant for *dry air*

$$R_d = \frac{k}{\sum_{i=N_2, O_2, Ar, \dots} f_i m_i} = \frac{k}{m_d}, \quad (3.7)$$

where the sum extends to all components other than water vapour, and the gas constant for water vapour

$$R_v = \frac{k}{m_v}. \quad (3.8)$$

3.2 Six ways to quantify moisture content

From the point of view of thermodynamics, the atmosphere may be considered a variable mixture of two components, dry air and water vapour. This mixture is called *moist air*. There are many equivalent and widely-used ways of specifying the amount of moisture in dry air. We list them here for later reference, together with the typical units:

Number density n_v [molecules m^{-3}]

Partial pressure $e = n_v kT$ [hPa]

Number fraction, also called volume mixing ratio, $f_v = n_v / (n_v + n_d) = e/p$ [%]

Mass density $\rho_v = n_v m_v$ [kg m^{-3}]

Specific humidity $q = \rho_v / \rho$ [g kg^{-1}]

Mass mixing ratio $w = \rho_v / \rho_d$ [g kg^{-1}]

Note the subtle distinction between specific humidity (mass density of water divided by total density of water+dry air mixture) and mass mixing ratio (density of water divided by density of dry air only).

The above definitions are all equivalent and can be expressed one as a function of the other. Some useful conversions are:

$$q = \frac{\rho_v}{\rho_v + \rho_d} = \frac{w}{w + 1}, \quad (3.9)$$

$$w = \frac{\rho_v}{\rho_d} = \frac{e/R_v T}{p_d/R_d T} = \frac{e}{p-e} \frac{R_d}{R_v} = \frac{\epsilon e}{p-e}, \quad (3.10)$$

and

$$f_v = \frac{q}{\epsilon - (\epsilon - 1)q}, \quad (3.11)$$

where

$$\epsilon = \frac{m_v}{m_d} = \frac{18}{28.9} = 0.622. \quad (3.12)$$

3.3 Potential temperature of moist unsaturated air

The potential temperature is defined as

$$\Theta = T \left(\frac{p}{p_0} \right)^{-R/c_p}, \quad (3.13)$$

where the R and c_p are *moist* values. Let us make the dependence on humidity explicit.

Firstly,

$$R = \frac{k}{f_d m_d + f_v m_v} = \frac{R_d}{1 + (\epsilon - 1)f_v}, \quad (3.14)$$

so using (3.11) we have

$$R = R_d \left(1 + \frac{1 - \epsilon}{\epsilon} q \right) = R_d (1 + 0.608q). \quad (3.15)$$

Because of equipartition, the heat capacity of a mixture is the sum of the heat capacities of the components. The specific heat capacity is just the mass-weighted mean:

$$c_p = c_{pd} \frac{\rho_d}{\rho} + c_{pv} \frac{\rho_v}{\rho} \quad (3.16)$$

where c_{pd} and c_{pv} are the specific heat capacities of dry air and water vapour (1005 and 1952 J K⁻¹ kg⁻¹ respectively). Thus

$$c_p = c_{pd} \left(1 - q + \frac{c_{pv}}{c_{pd}} q \right) = c_{pd} (1 + 0.94q), \quad (3.17)$$

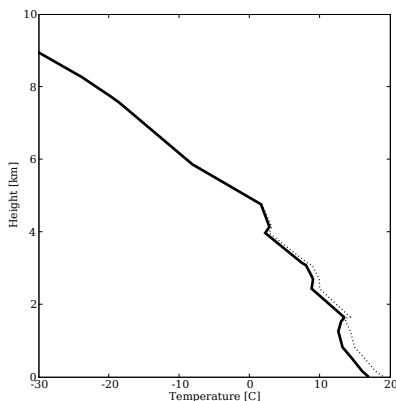


Figure 3.1: Sounding at Valentia showing temperature (solid) and virtual temperature (dotted).

and

$$\frac{R}{c_p} = \frac{R_d}{c_{pd}} \left(\frac{1 + 0.608q}{1 + 0.94q} \right) \simeq \frac{R_d}{c_{pd}} (1 - 0.33q) \quad (3.18)$$

Since q is rarely larger than about 4% in the atmosphere, the ratio R/c_p does not deviate from the dry value by more than 1%, and this difference is safely ignored. Thus, even for moist air we take

$$\Theta = T \left(\frac{p}{p_0} \right)^{-R_d/c_{pd}}. \quad (3.19)$$

3.4 Virtual temperature

The ideal gas law, $p = nkT$, says that pressure depends on the number of molecules but not on their mass. This is somewhat counter-intuitive, since we might expect heavier molecules to bang more strongly on the sides of the box and produce a higher pressure. The trick is that temperature is the product of mass and mean square velocity: for a given temperature, lighter molecules travel faster and produce the same momentum flux. As a result, at fixed temperature and pressure air becomes less dense the moister it is (since $m_v < m_d$).

A quirk of meteorology is an insatiable desire to express everything in terms of a tempera-

ture: thus, entropy is expressed as a *potential* temperature, the density effect of moisture is expressed as a *virtual* temperature, and we'll encounter a few more as go along; they are all unknown outside meteorology. To define virtual temperature, we use (3.15) above to write the ideal gas law for moist air as

$$p = \rho R_d T_v \quad (3.20)$$

where the virtual temperature is defined by

$$T_v = (1 + 0.608q)T. \quad (3.21)$$

Virtual temperature can be up to 2–3% higher than ordinary temperature, which can mean a difference of several degrees (Fig. 3.1).

3.5 Static stability of moist non-condensing air

We can take temperature madness further by defining a *virtual potential* temperature

$$\Theta_v = T_v \left(\frac{p}{p_0} \right)^{-R_d/c_{pd}}, \quad (3.22)$$

using dry values in the exponent as discussed in the Section 3.3. The beauty of virtual temperature is that we can now carry through the derivation of the Brunt-Väisälä frequency exactly as in Section 2.22, simply replacing Θ with Θ_v everywhere (you should go through the derivation and think about why this is possible). Thus, in moist, unsaturated air the static stability criterion is

$$\frac{d\Theta_v}{dz} > 0 \quad (\text{stable}), \quad \frac{d\Theta_v}{dz} < 0 \quad (\text{unstable}). \quad (3.23)$$

As can be seen by explicitly computing the vertical derivatives, this implies

$$\frac{dT_v}{dz} > -\frac{g}{c_{pd}} \quad (\text{stable}), \quad \frac{dT_v}{dz} < -\frac{g}{c_{pd}} \quad (\text{unstable}). \quad (3.24)$$

Since virtual temperature is always greater than temperature, and moisture is always greater near the surface, the virtual temperature effect always acts to *destabilise* the atmosphere.

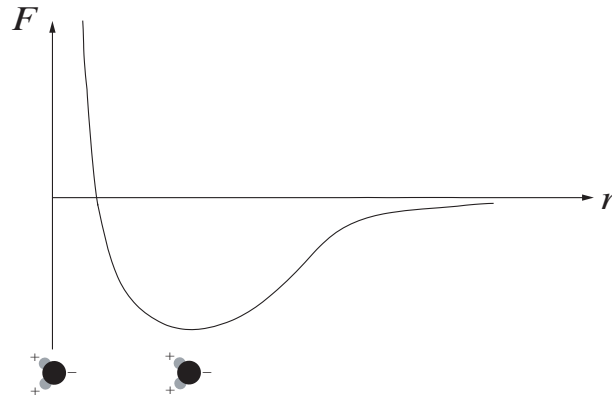


Figure 3.2: Schematic of the intermolecular force between two water molecules.

3.6 Inter-molecular forces

A key part of the definition of an ideal gas is that the component molecules do not interact with each other. This is not true in reality: the molecules of all gases can exert strong electrostatic forces on one another (Fig. 3.2). When two molecules are very close, they will repel each other, but when they are somewhat farther apart they will attract. The forces drop off to zero quite rapidly. If the gas is dilute, the molecules will on average be far from each other, the intermolecular forces can be neglected and the gas will behave as an ideal gas; this is the case for the atmosphere. However, if density is high enough, molecules will spend more time close to each other, intermolecular attraction will play a greater role and the gas may condense. Whether or not this happens depends on the mean separation between molecules (i.e. density) and the mean speed of the molecules (i.e. temperature): fast-moving molecules will escape each other's attraction, just as they may escape Earth's gravity (Section 2.10). Because of its peculiar arrangement of electrons and nuclei, the water molecule has a strong, permanent electric dipole and the attractive force is very strong. For this reason, water can condense at the typical temperature and (partial) pressure at which it is found in the atmosphere, unlike other major constituents which do not have permanent dipoles.

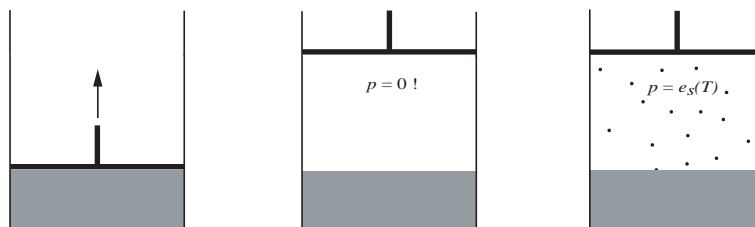


Figure 3.3: A cylinder with a tight-fitting piston containing liquid water. When the piston is lifted, water molecules leave the liquid and fill the empty space until their density is such that the flux of outgoing molecules matches the flux of molecules coming back in to the liquid.

3.7 Saturation vapour pressure

Consider the thought experiment illustrated in Fig. 3.3. We put some liquid water in a cylinder and cap it with a perfectly-fitting piston, with no air between the piston and the cylinder. Now we yank the piston upwards. If the cross-section of the cylinder is small enough, you can do this with your hand. What happens next? In the first instant, there will be a vacuum between the water and the piston. But this will not last long: water molecules will soon escape from the liquid and fill the cavity with vapour. This process is very much like the escape-from-gravity process of Section 2.10. In the liquid, the mean distance between molecules is roughly where the force curve in Fig. 3.2 crosses zero. Just as in the gas phase, the molecules are jiggling around—they have a mean KE, measured by the temperature of the liquid. As two molecules move apart, the intermolecular force does work on them, resulting in conversion of KE into electrostatic PE: the molecules slow down, and eventually move back together again. However, molecular velocities are again distributed according to the Maxwell-Boltzmann distribution, and there will be a finite number of molecules with sufficient KE to entirely overcome the attraction and escape to infinity, joining the gas phase.

As molecules fill the cavity and bounce around in it, some of them will re-enter the liquid. Eventually, a steady state will be reached where the flux of particles leaving the liquid matches the ingoing flux. The pressure in the cavity at this point is called the *saturation vapour pressure* and given the symbol e_s . The saturation vapour pressure clearly depends on temperature, and will increase with increasing temperature. It also depends on the strength

of the intermolecular forces, and will be greater for liquids with weaker intermolecular attraction. As we will see later, it also depends on the shape of the liquid-gas interface and on the purity of the liquid. Thus we can state that *the saturation vapour pressure of water above a plane surface of pure liquid depends only on temperature, $e_s = e_s(T)$.*

3.8 Relative humidity and dew-point temperature

In Section 3.2 we gave 6 ways of specifying the humidity content of air. The concept of saturation vapour pressure allows for two more definitions:

- *Relative humidity.* Given the saturation vapour pressure e_s , we can use (3.10) to define the saturation mixing ratio

$$w_s = \frac{\epsilon e_s}{p - e_s}. \quad (3.25)$$

Relative humidity r is then defined as the ratio of the actual mixing ratio to the saturation value:

$$r = \frac{w}{w_s}, \quad (3.26)$$

which again using (3.10) can be written

$$r = \frac{e}{e_s} \frac{p - e_s}{p - e} \simeq \frac{e}{e_s}, \quad (3.27)$$

since e/p is never more than a few percent.

- *Dew point temperature T_d* is defined as the temperature to which an air parcel must be cooled at constant pressure to achieve saturation, and is given by

$$e = e_s(T_d). \quad (3.28)$$

3.9 Latent heat of vaporisation

It is clear from the discussion above that energy transformations play a key role in evaporation and condensation. Only the most energetic molecules can leave the liquid, and so evaporation implies a net loss of KE for the liquid and a consequent lowering of the temperature: this is why sweating cools you down. In exactly the reverse process, the molecules accelerate as they enter the fluid, attaining above-average KE and hence increasing its temperature: this is the famous “release of latent heat”.

How can we quantify the energies involved? Consider the situation in the right-most panel in Fig. 3.3, but now imagine that the outside pressure is $e_s(T)$, and that the cylinder is adiabatic. Now add a little heat to the liquid (how you do this through an adiabatic container is part of the magic of thought experiments): instead of raising its temperature, the heat will go into evaporating some of the liquid. The energy required to evaporate 1 kg of liquid under these conditions is called the *latent heat of vaporisation*, and is given by

$$\ell_v = u_v - u_l + p(\alpha_v - \alpha_l). \quad (3.29)$$

The term $u_v - u_l$ is the difference in internal energy (per unit mass) between the vapour and liquid: it is the total energy required to overcome the molecular attractions between 1 kg’s worth of molecules. The second term is the work done against the external pressure to expand the cavity and keep it at constant pressure: $\alpha_v - \alpha_l$ is the change in volume when 1 kg of liquid water is converted into vapour at constant pressure. Recalling that the enthalpy per unit mass is

$$h = u + p\alpha, \quad (3.30)$$

we can also write

$$\ell_v = h_v - h_l. \quad (3.31)$$

Calculating ℓ_v from first principles, assuming a given structure for intermolecular forces, is a very difficult task. Fortunately, we don’t need to do it: we can just *measure* its value. It turns out to be about $2.5 \times 10^6 \text{ J kg}^{-1} \text{ K}^{-1}$ (at 0°C). This is a lot of energy: the heat capacity

of liquid water is $4218 \text{ J kg}^{-1} \text{ K}^{-1}$, so with the heat released by condensing 1 kg of water vapour you could bring more than 7 kg of water from room temperature to the boiling point.

3.10 Wet-bulb temperature

Consider a drop of rain falling in a column of *unsaturated* air with uniform temperature and humidity. Let's assume that the drop's temperature is initially the same as the surroundings'. Since the surroundings are unsaturated, the drop cools by evaporation as it falls, but is warmed by contact with the air. Eventually, the drop will reach a steady-state temperature. What is this temperature?

To answer this, consider the situation once the drop has reached its steady-state temperature, which we will call T_w . As the drop falls into some new, undisturbed air, it will find itself out of equilibrium with the air around it. The drop's surface will quickly equilibrate with a very thin layer of air surrounding it (the thickness of this layer will be a few times the distance a molecule can travel before colliding with another molecule, typically $1 \mu\text{m}$ or less). Here, "equilibrate" means that by exchanging energy and molecules, the drop's surface and the thin air layer quickly reach the same temperature, and the air becomes saturated. However, we are *assuming* that the drop's temperature is in steady state, so after equilibration the surface of the drop will have come back to temperature T_w . Thus the air around the molecule will *also* have temperature T_w , and its vapour pressure will be the saturation vapour pressure at T_w . As the drop moves on, the saturated layer is stripped away and replaced by fresh unsaturated air, and the whole process is repeated. Thus the drop continuously loses mass, leaving behind itself a trail of saturated air at T_w .

Based on this picture, we can see that T_w is *the temperature to which an air parcel drops when it gives up the energy required to evaporate just enough water to bring it to saturation*. T_w is called the *wet-bulb temperature*. To make the definition precise, we make two further assumptions: that the process occurs at constant pressure, and that the water which is being

evaporated is already at temperature T_w . We can now write down an equation relating wet-bulb temperature to temperature and mixing ratio. The amount of energy (more precisely, enthalpy) needed to raise the temperature of a moist parcel from T_w to T at constant pressure is

$$(M_d c_{pd} + M_v c_{pv})(T - T_w), \quad (3.32)$$

where M_d and M_v are the masses of dry air and moisture in the parcel. The amount of enthalpy needed to evaporate enough water to make the parcel saturated is

$$\ell_v(M_{vs} - M_v) \quad (3.33)$$

where M_{vs} is the mass of water in the parcel when it is saturated at the wet-bulb temperature. Setting (3.32) equal to (3.33) and dividing by M_d gives

$$(c_{pd} + w c_{pv})(T - T_w) = \ell_v(w_s - w). \quad (3.34)$$

Neglecting the $w c_{pv}$ contribution on the l.h.s., we have

$$w = w_s(T_w) - \frac{c_{pd}}{\ell_v}(T - T_w), \quad (3.35)$$

or using $w \simeq \epsilon e/p$,

$$e = e_s(T_w) - \frac{p c_{pd}}{\epsilon \ell_v}(T - T_w) \quad (3.36)$$

which is known as the *psychrometric equation*.

If we have a good way of measuring T_w , we have a way to determine the humidity of air (“psychro” is Greek for “cold”, so “psychrometric” means “measuring the cold”). Humidity is tricky to measure directly, but measuring the temperature is easy. In a *sliding psychrometer*, the bulb of a thermometer is wrapped in gauze soaked in water and then spun around; this approximates the situation for the falling raindrop, and the steady-state temperature reading approximates T_w for the ambient air. From a forecasting point of view, if T_w falls below zero in a layer near the surface, then snow or hail falling through the layer will reach the surface without melting, while rain falling through this layer may reach the surface as freezing rain—supercooled drops that freeze on impact.

3.11 The Clausius-Clapeyron equation

We saw above that the saturation vapour pressure above a flat surface of pure liquid depends only on temperature. The Clausius-Clapeyron equation quantifies this relationship. It can be derived from very general thermodynamic considerations. The derivation is somewhat involved and not amazingly illuminating from a physical point of view, so we will not give it here; it can be found in many atmospheric science textbooks, including Bohren & Albrecht and Wallace & Hobbs. The Clausius-Clapeyron equation states

$$\frac{de_s}{dT} = \frac{1}{T} \frac{\ell_v}{\alpha_v - \alpha_l} \quad (3.37)$$

where ℓ_v is the latent heat of vaporisation and $\alpha_v - \alpha_l$ is the change in specific volume upon vaporisation. Note that the term on the r.h.s. is always positive, so saturation vapour pressure always increases with temperature.

The specific volume of liquid water is always much smaller than that of the vapour, so we can approximate

$$\frac{de_s}{dT} \simeq \frac{1}{T} \frac{\ell_v}{\alpha_v} = \frac{e_s \ell_v}{R_v T^2}. \quad (3.38)$$

If we assume ℓ_v is a constant (which it isn't), this can be integrated to give

$$e_s(T) = e_{s0} \exp\left(\frac{\ell_v}{R_v T_0}\right) \exp\left(-\frac{\ell_v}{R_v T}\right). \quad (3.39)$$

where T_0 is a reference temperature and e_{s0} the corresponding vapour pressure, which must be empirically determined: for reference, $T_0 = 0^\circ\text{C}$ gives $e_{s0} = 6.11$ hPa.

Equation 3.39 has a nice physical interpretation: writing

$$\frac{\ell_v}{R_v T} = \frac{m_v \ell_v}{kT}, \quad (3.40)$$

we see that the exponent is the ratio of the energy required to evaporate a single molecule of liquid, to the mean kinetic energy of the molecules.

For more accurate work, we need to include the temperature dependence of ℓ_v . From (3.29) we see that

$$\frac{d\ell_v}{dT} = c_{pv} - c_l, \quad (3.41)$$

the difference in specific heats of vapour and liquid (note that since the volume of liquid water changes very little with temperature, the specific heat at constant volume and at constant pressure are almost identical, so we use the single symbol c_l to indicate both). The difference between specific heats *is* to a good approximation constant with temperature (at least over the range of interest to Earth's atmosphere), so we can integrate to obtain

$$\ell_v = \ell_{v0} + (c_{pv} - c_l)(T - T_0) \quad (3.42)$$

where ℓ_{v0} is the latent heat at some reference temperature T_0 . Substituting this into (3.38) and integrating:

$$\ln \frac{e_s}{e_{s0}} = \frac{\ell_{v0} + (c_l - c_{pv})T_0}{R_v} \left(\frac{1}{T_0} - \frac{1}{T} \right) - \frac{c_l - c_{pv}}{R_v} \ln \frac{T}{T_0} \quad (3.43)$$

$$= 6808 \left(\frac{1}{T_0} - \frac{1}{T} \right) - 5.09 \ln \frac{T}{T_0}, \quad (3.44)$$

where the second equality is obtained by taking $T_0 = 0^\circ\text{C}$.

3.12 Scale height of water vapour

Since temperature decreases with height in the troposphere, e_s will also decrease with height, and we expect the atmosphere to become drier with height. We can estimate the rate of decrease by assuming an atmosphere with a dry-adiabatic lapse rate, so that using (3.39):

$$e_s(z) = e_{s0} \exp\left(\frac{\ell_v}{R_v T_0}\right) \exp\left(-\frac{\ell_v}{R_v(T_0 - \Gamma_d z)}\right) \simeq e_{s0} \exp\left(-\frac{z}{H_v}\right), \quad (3.45)$$

where the *water vapour scale height*

$$H_v = \frac{R_v T_0^2}{\ell_v \Gamma_d} = \frac{R_v T_0}{g} \frac{c_{pd} T_0}{\ell_v} = \frac{c_{pd} T_0}{\epsilon \ell_v} H, \quad (3.46)$$

with $H = R_d T_0 / g$ the pressure scale height for dry air and ϵ given by (3.31). For Earth-like parameter values, $H_v \sim H/5 \sim 2$ km.

3.13 Level of cloud formation: the lifting condensation level

The most common way to form clouds on Earth is by lifting: as moist air rises, it cools and eventually becomes saturated, at which point a cloud forms. Section 2.3.1 reviews lifting mechanisms in the atmosphere; these mechanisms are generally rapid enough that air parcels are lifted adiabatically. The level at which a parcel adiabatically lifted from near the surface first reaches saturation is called the *lifting condensation level* or LCL. If the parcel is lifted further, a cloud forms. On a sunny summer day, strong solar heating at the surface produces dry static instability; the consequent rising motion can produce clouds known as *fair weather cumulus*, whose sharply-defined base corresponds to the LCL.

To estimate the height of the LCL, note that the number fraction of water molecules in the parcel, f_v , remains constant as the parcel is lifted, so

$$e(z) = f_v p(z) \simeq f_v p_0 \exp(-z/H). \quad (3.47)$$

Thus the vapour pressure in the parcel decreases at a rate given by the pressure scale height. The *saturation* vapour pressure, on the other hand, decreases at the much faster rate given by the water vapour scale height. The LCL is where the two curves meet and $e = e_s$ (Fig. 3.4). Setting (3.47) equal to (3.45) gives

$$z_{\text{LCL}} = \frac{HH_v}{H - H_v} \ln \frac{e_{s0}}{f_v p_0} \simeq -H_v \ln r_0 \quad (3.48)$$

using $H \gg H_v$; here r_0 is the relative humidity of the parcel before lifting. If $r_0 = 0$ then $z_{\text{LCL}} = \infty$ and the parcel never saturates; if $r_0 = 1$ then $z_{\text{LCL}} = 0$ and the parcel is already saturated at the ground. For typical near-surface r values of 70-80%, $z_{\text{LCL}} \sim 700$ m.

Note carefully that cloud formation on ascent is possible on Earth only by virtue of the fact that $H > H_v$. The opposite case, $H < H_v$, is entirely feasible; as shown by (3.46), it requires higher temperature and/or higher c_{pv}/ℓ_v , which is possible in planetary atmospheres. In this case, clouds would form on *descent*. You should look at Fig. 3.4 and convince yourself of this.

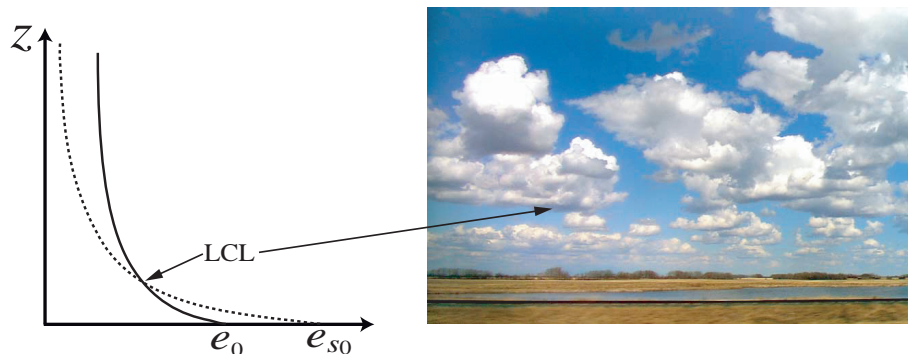


Figure 3.4: Vapour pressure (solid) and saturation vapour pressure (dotted) for a parcel adiabatically lifted from the ground where $e = f_v p_0 = e_0$ and $e_s = e_{s0}$. Also shown is a field of fair weather cumulus clouds.

3.14 Moist entropy and equivalent potential temperature

In complete analogy to what we did in Section 2.16, we will now derive an expression for the entropy of a moist air parcel in which condensation may occur. This will allow us to define a moist equivalent of the dry potential temperature. We begin with the law of conservation of energy:

$$\frac{dU}{dt} = Q - p \frac{dV}{dt}. \quad (3.49)$$

For a saturated air parcel, the internal energy can be written as

$$U = M_d u_d + M_v u_v + M_l u_l \quad (3.50)$$

where subscripts d , v and l indicate dry air, water vapour and liquid water components respectively and M_i is the mass of component i . Thus

$$\frac{dU}{dt} = M_d \frac{du_d}{dt} + M_v \frac{du_v}{dt} + M_l \frac{du_l}{dt} + (u_v - u_l) \frac{dM_v}{dt}, \quad (3.51)$$

since $dM_v/dt = -dM_l/dt$ (i.e., the increase of vapour mass is equal to the loss of liquid mass). Analogously, the volume V of the parcel can be written

$$V = V_l + (V - V_l) = M_l \alpha_l + M_v \alpha_v = M_l \alpha_l + M_d \alpha_d, \quad (3.52)$$

where V_l is the volume occupied by liquid droplets and $\alpha = 1/\rho = V/M$ is the specific volume; Eq. (3.52) essentially states that both dry air and water vapour occupy the same volume, $V - V_l$. Differentiating (3.52) and multiplying by p gives

$$p \frac{dV}{dt} = p \left(M_l \frac{d\alpha_l}{dt} \Big|_{M_l} + M_v \frac{d\alpha_v}{dt} \Big|_{M_v} + \alpha_v \frac{dM_v}{dt} \Big|_{\alpha_v} + \alpha_l \frac{dM_l}{dt} \Big|_{\alpha_l} \right) \quad (3.53)$$

$$= p \left(M_v \frac{d\alpha_v}{dt} + (\alpha_v - \alpha_l) \frac{dM_v}{dt} \right) \quad (3.54)$$

$$= p_d M_d \frac{d\alpha_d}{dt} + p_v M_v \frac{d\alpha_v}{dt} + p(\alpha_v - \alpha_l) \frac{dM_v}{dt}. \quad (3.55)$$

Equation (3.54) follows by taking

$$M_l \frac{d\alpha_l}{dt} \Big|_{M_l} \approx 0, \quad (3.56)$$

since the volume of a fixed mass of liquid is essentially constant, and by again using $dM_v/dt = -dM_l/dt$. Equation (3.55) follows by writing p as the sum of partial pressures, $p = p_d + p_v$, and noting that

$$M_v \frac{d\alpha_v}{dt} \Big|_{M_v} = \frac{d}{dt}(V - V_l) = M_d \frac{d\alpha_d}{dt} \Big|_{M_d}. \quad (3.57)$$

Substituting in (3.49), we obtain

$$(M_d c_{vd} + M_v c_{vv} + M_l c_l) \frac{dT}{dt} + \ell_v \frac{dM_v}{dt} + p_d M_d \frac{d\alpha_d}{dt} + p_v M_v \frac{d\alpha_v}{dt} = Q \quad (3.58)$$

with the latent heat of vaporisation ℓ_v given by (3.29).

We now write

$$p_d M_d \frac{d\alpha_d}{dt} = M_d R_d \left(\frac{dT}{dt} - \frac{T}{p_d} \frac{dp_d}{dt} \right) \quad (3.59)$$

and

$$p_v M_v \frac{d\alpha_v}{dt} = M_v R_v \left(\frac{dT}{dt} - \frac{T}{e} \frac{de}{dt} \right) \quad (3.60)$$

Substituting into (3.58) and dividing by $M_d T$ we obtain

$$(c_{pd} + w_s c_{pv} + w_l c_l) \frac{1}{T} \frac{dT}{dt} - \frac{R_d}{p_d} \frac{dp_d}{dt} - \frac{w R_v}{e} \frac{de}{dt} + \frac{\ell_v}{T} \frac{dw}{dt} = \frac{Q}{M_d T}. \quad (3.61)$$

where w is the vapour mixing ratio and w_l is the liquid water mixing ratio.

Now

$$\frac{\ell_v}{T} \frac{dw}{dt} = \frac{1}{T} \frac{d}{dt}(\ell_v w) - w(c_{pv} - c_l) \frac{1}{T} \frac{dT}{dt} \quad (3.62)$$

$$= \frac{d}{dt} \left(\frac{\ell_v w}{T} \right) + \frac{\ell_v w}{T^2} \frac{dT}{dt} - w(c_{pv} - c_l) \frac{1}{T} \frac{dT}{dt} \quad (3.63)$$

where we have used $d\ell_v/dT = c_{pv} - c_l$. Substituting in (3.61) and using Clausius-Clapeyron gives

$$[c_{pd} + (w + w_l)c_l] \frac{1}{T} \frac{dT}{dt} - \frac{R_d}{p_d} \frac{dp_d}{dt} - \frac{wR_v}{e} \frac{de}{dt} + \frac{wR_v}{e_s} \frac{de_s}{dt} + \frac{d}{dt} \left(\frac{\ell_v w}{T} \right) = \frac{Q}{M_d T}. \quad (3.64)$$

Defining the effective heat capacity

$$c_p = c_{pd} + (w + w_l)c_l, \quad (3.65)$$

(which is a constant, since total water is conserved), we see that the *moist entropy*

$$s = c_p \ln \left[T p_d^{-R_d/c_p} \left(\frac{e}{e_s} \right)^{-wR_v/c_p} \exp \left(\frac{\ell_v w}{c_p T} \right) \right] + const. \quad (3.66)$$

is conserved under reversible adiabatic transformations. The *equivalent potential temperature*,

$$\Theta_e = T \left(\frac{p_d}{p_0} \right)^{-R_d/c_p} \left(\frac{e}{e_s} \right)^{-wR_v/c_p} \exp \left(\frac{\ell_v w}{c_p T} \right), \quad (3.67)$$

is also conserved. Physically, Θ_e is the temperature a saturated parcel would have if all the water vapour in it were to condense and the parcel were brought to sea level. Note that the above definition of Θ_e applies whether or not the parcel is saturated. However, the derivation assumes that the parcel is in thermodynamic equilibrium at all times, so *if the parcel is subsaturated there can be no liquid water*, $w_l = 0$ (any liquid water in a subsaturated parcel will evaporate irreversibly until the parcel is saturated). Note also that if the parcel is perfectly dry, then $\Theta_e = \Theta$, the dry potential temperature (note that $\lim_{x \rightarrow 0} x^x = 1$).

For a subsaturated parcel, we can also define a *saturation equivalent potential temperature* Θ_{es} as the equivalent potential temperature that an unsaturated parcel would have *if it were saturated*:

$$\Theta_{es} = T \left(\frac{p_d}{p_0} \right)^{-R_d/c_p} \exp \left(\frac{\ell_v(T)w_s(T)}{c_p T} \right), \quad (3.68)$$

with $c_p = c_{pd} + w_s(T)c_l$, where $w_s(T)$ is the saturation mixing ratio at temperature T . Note that Θ_{es} is *not* conserved as a parcel is adiabatically lifted.

3.15 The moist adiabatic lapse rate

In Section 2.20 we derived the dry adiabat, defined as the temperature profile a non-condensing atmosphere needs to have in order for the temperature in an adiabatically-lifted parcel always to match that of its surroundings. Here we derive the analogous result for a parcel in which condensation is occurring.

For a saturated parcel lifted adiabatically at speed dz/dt , (3.64) implies

$$c_p \frac{dT}{dz} - \frac{R_d T}{p_d} \frac{dp_d}{dz} + T \frac{d}{dz} \left(\frac{\ell_v w_s}{T} \right) = 0. \quad (3.69)$$

Now let's work on the 2nd term:

$$\frac{R_d T}{p_d} \frac{dp_d}{dz} = \frac{1}{\rho_d} \frac{d}{dz} (p - e_s) = -\frac{\rho'}{\rho_d} g - \frac{\rho_v}{\rho_d} \frac{de_s}{dT} \frac{dT}{dz} = -\frac{p R_d T}{p_d R' T'} g - \frac{\ell_v w_s}{T} \frac{dT}{dz}. \quad (3.70)$$

We have used Clausius-Clapeyron in the last step; primes refer to properties of the *surroundings* (we do not use subscript s , as in Section 2.20, to avoid confusion with s for “saturated” as used here), which are assumed hydrostatic. As for the dry case, we define the moist adiabat as the temperature profile an atmosphere needs to have so that temperature within an adiabatically-lifted saturated parcel always matches the surroundings, $T = T'$. This still leaves an annoying $p R_d / p_d R'$ in (3.70) which we will simply approximate as 1, since moisture mixing ratio never exceeds a few percent. With this approximation, (3.69) and (3.70) give

$$\frac{dT}{dz} = -\frac{g}{c_p} - \frac{1}{c_p} \frac{d}{dz} (\ell_v w_s). \quad (3.71)$$

Thus the moist-adiabatic lapse rate is simply the dry-adiabatic lapse rate plus a contribution due to condensation. Note that since

$$w_s = \frac{\epsilon e_s}{p - e_s} \simeq \frac{\epsilon e_s}{p} \sim \frac{\epsilon e_{s0}}{p_0} \exp \left(-\frac{z}{H_v} + \frac{z}{H} \right) \quad (3.72)$$

and $H_v < H$ (see Section 3.12), w_s decreases exponentially with height, so the condensation term in (3.71) is positive. This means that *the moist adiabatic lapse rate is always less than the dry adiabatic*. To see this more explicitly, we can use the approximation in (3.72) to write

$$\frac{1}{w_s} \frac{dw_s}{dz} = -\frac{1}{p} \frac{dp}{dz} + \frac{1}{e_s} \frac{de_s}{dz} = \frac{g}{RT} + \frac{\ell_v}{R_v T^2} \frac{dT}{dz}. \quad (3.73)$$

Taking ℓ_v as constant in (3.71) and using (3.73) finally gives

$$\frac{dT}{dz} = -\frac{g}{c_{pd} + (w_s + w_l)c_l} \frac{1 + \ell_v w_s / RT}{1 + \ell_v^2 w_s / c_{pd} R_v T^2}. \quad (3.74)$$

Thus the moist adiabatic lapse rate will be less than the dry adiabatic if $\ell_v w_s / RT < \ell_v^2 w_s / c_{pd} R_v T^2$, which implies $c_{pd} T / \epsilon \ell_v < 1$: as we saw in Section 3.12, this is true in our atmosphere.

3.16 Moist adiabats and pseudoadiabats

To compute the structure of the moist adiabat explicitly, we need to integrate (3.74) in the vertical. Because of the complicated dependence of w_s on T , this needs to be done numerically. In practise, we use a finite-difference approximation to write

$$T(z + \Delta z) = T(z) - \left(\frac{g}{c_{pd} + (w_s + w_l)c_l} \frac{1 + \ell_v w_s / RT}{1 + \ell_v^2 w_s / c_{pd} R_v T^2} \right) \Big|_z \Delta z, \quad (3.75)$$

which allows the profile to be built up step by step given initial values of temperature and humidity. To actually do the computation, we need to express w_s , ℓ_v , and w_l as functions of z :

- For ℓ_v , we use (3.42) to write $\ell_v(z) = \ell_v(T(z))$.
- For w_s we take

$$w_s(z) = \frac{\epsilon e_s(z)}{p(z) - e_s(z)} \quad (3.76)$$

where $e_s(z) = e_s(T(z))$. $p(z)$ is computed using the hydrostatic equation, which consistently with the approximation $\rho' \simeq \rho_d$ made to derive (3.71) can be written

$$\frac{d \ln p}{dz} = -\frac{g}{R_d T} \quad (3.77)$$

and integrated numerically as above.

- For w_l (the mass mixing ratio of liquid water), the situation is more complicated. If none of the condensed water precipitates out of the parcel during ascent, then the total water content $w_t = w_s + w_l$ is constant, in which case we simply use the initial value of w_t in (3.75). This is called a *true adiabat* or *reversible adiabat*, in which entropy (and equivalent potential temperature) is exactly conserved. In realistic situations, some of the condensed water may fall out as precipitation. Exactly how much falls out depends on somewhat intractable cloud microphysical processes, discussed in Chapter 4. If *all* the condensate drops out, then we can set $w_l = 0$ in (3.75). This yields the *pseudoadiabat*; “pseudo” because it’s not a real adiabat, since entropy is not exactly conserved.

Some examples are shown in Fig. 3.5. To compute each of these curves, we start with a saturated parcel of specified temperature at the surface and integrate upwards. As the parcel rises, water condenses releasing latent heat, and so temperature decreases more slowly than in the dry case. This effect is stronger the warmer (and hence moister) the initial conditions. At typical Earth-like surface temperatures, the effect is very strong: for a starting temperature of 5°C, the mean lapse rate over the first 5 km is 7.2 K km⁻¹, for 20°C it is 4.8 °C km⁻¹ and for 35°C it is 3.3 °C km⁻¹ (compare with 9.8 °C km⁻¹ for the dry adiabat).

As the parcel rises, more and more water condenses and the effect on the lapse rate becomes weaker. At great height, temperatures are very low, w_s becomes very small, and the pseudoadiabatic lapse rate converges to the dry adiabatic, g/c_{pd} , while the true adiabatic lapse rate converges to the smaller value $g/(c_{pd} + w_t c_l)$.

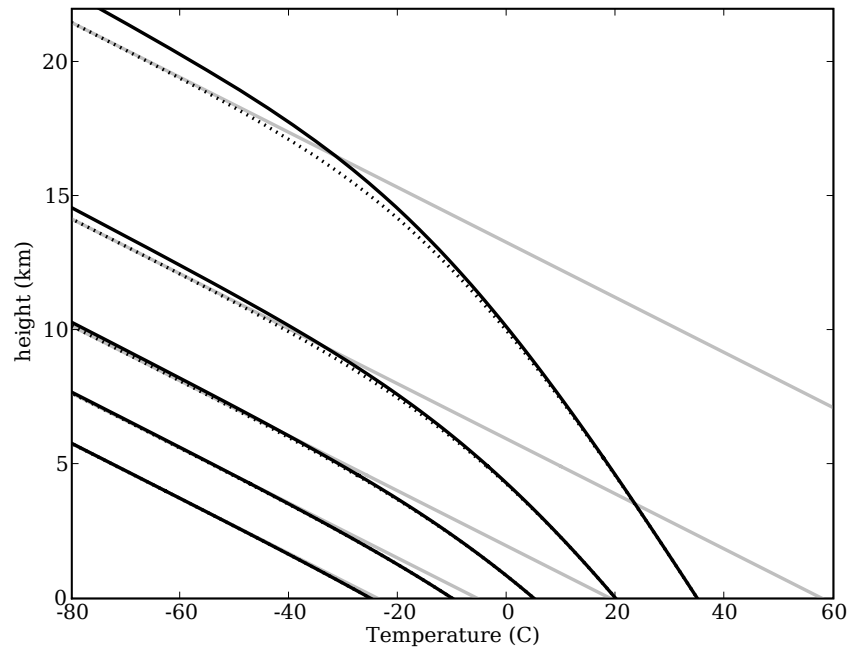


Figure 3.5: Moist adiabats (black solid lines) and pseudoadiabats (dotted) starting at $z = 0$ with temperatures of -25 , -10 , 5 , 20 and 35°C and the corresponding saturation humidity, with no condensed water initially. Surface pressure is 1000 hPa. Gray lines show dry adiabats to which the pseudoadiabats converge at high altitude.

3.17 Visualising the connection between the various meteorological temperatures

As we have seen, meteorologists enjoy defining a bewildering array of temperatures (potential, equivalent, wet-bulb etc.) which are connected to everyday absolute temperature by well-defined physical processes. These processes can be visualised graphically as shown in Fig. 3.6, which provides a handy way to tie all the temperatures together and remember the processes that connect them.

Consider a parcel initially at some height above the surface (black dot in Fig. 3.6), where

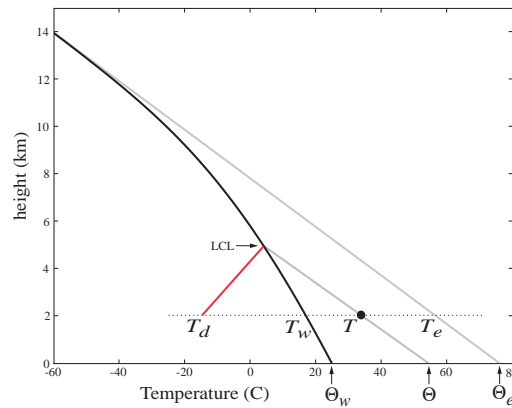


Figure 3.6: Transformations of an unsaturated parcel (black dot) lifted or lowered adiabatically from an initial state with temperature T . Black solid line is a moist pseudoadiabat. Gray lines are dry adiabats. Red line shows change in dew point temperature.

it has some temperature T , some pressure p and some mixing ratio $w < w_s$ (i.e. it is unsaturated). If the parcel is lowered adiabatically, it will follow a dry adiabat. When it reaches the surface, its temperature will equal its potential temperature Θ (*potential temperature is the temperature an unsaturated parcel would have if brought adiabatically to the surface*). If the parcel is raised adiabatically, it will follow a dry adiabat up to the LCL. At this point the parcel is saturated, and upon further lifting will follow a moist adiabat. If all condensate is removed from the parcel, then it follows a pseudoadiabat. Following a pseudoadiabat all the way to the top of the atmosphere results in complete drying of the parcel. If we then bring the parcel down again, it will follow a dry adiabat. When we get back down to the initial height, the parcel's temperature will equal the equivalent temperature T_e (*equivalent temperature is the temperature a parcel would have if all its water were made to condense while pressure was kept fixed*). If we keep going to the surface, the parcel's temperature will be the equivalent potential temperature Θ_e (*equivalent potential temperature is the temperature a parcel would have if all its water were made to condense and the parcel were brought adiabatically to the surface*).

Now consider the final possibility: we raise the parcel dry-adiabatically from its initial level to the LCL, and then *lower* it pseudoadiabatically. In pseudoadiabatic descent, water is *added* to the parcel—just enough water to keep the parcel saturated (exactly the opposite to

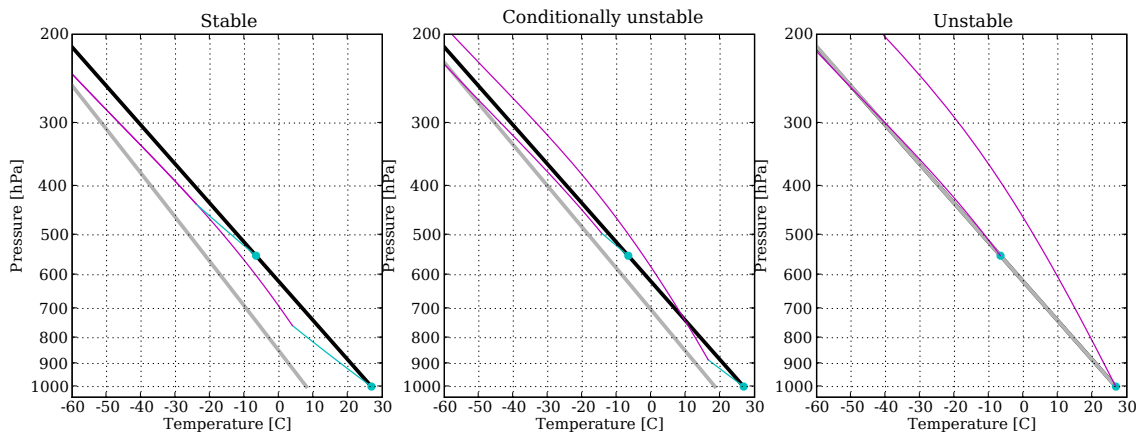


Figure 3.7: Three idealised temperature (black line) and dew-point (gray) soundings, all with the same fixed temperature lapse rate of $7^{\circ}\text{C}/\text{km}$ and vertically constant relative humidity of 30% (left), 60% (middle) and 100% (right).

the removal of condensate in pseudoadiabatic ascent). When we get back to the initial level, the parcel's temperature will be the wet-bulb temperature T_w (*wet-bulb temperature is the temperature a parcel reaches by evaporating enough water to make itself saturated*). Finally, if we keep going to the surface (always following a pseudoadiabat) we reach the wet-bulb potential temperature Θ_w .

3.18 Static stability of a moist atmosphere

In Section 2.21, we examined the stability of a dry atmosphere to *infinitesimal* displacements of a test parcel. This gives two stability categories: if a parcel displaced upward becomes positively buoyant, then the temperature profile is unstable; otherwise, it is stable. When dealing with the stability of a moist atmosphere, it is useful to consider also *finite-size* displacements. This introduces a third category, called *conditional instability*: a profile is unstable if a parcel can become positively buoyant when displaced *far enough* upward

Consider for instance the three idealised soundings depicted in Fig. 3.7. In all 3 cases,

the ambient lapse rate is $7^{\circ}\text{C km}^{-1}$ and relative humidity is constant with height. In the left panel, relative humidity is 30%. A parcel lifted from the surface, being unsaturated, will initially follow a dry adiabat. Eventually, it will reach its LCL and then follow a moist adiabat. Throughout, its temperature will be less than the surroundings', and the parcel will be negatively buoyant (i.e. will want to fall back down); the same is true for parcels lifted from any level. Thus the profile is stable to parcel displacements *of any size* from any level.

In the middle panel, relative humidity is 60%. A parcel lifted from the surface will initially be negatively buoyant. But some distance above the LCL, at a point called the *level of free convection* (LFC), the moist adiabat crosses the sounding temperature, and the parcel becomes positively buoyant: this is an example of conditional instability (which in turn is an example of *subcritical instability*, the general term for instabilities requiring a triggering perturbation of finite size). Parcels lifted from higher up in the atmosphere, on the other hand, never achieve positive buoyancy. Overall, the profile is stable to all infinitesimal perturbations but unstable to some finite-size perturbations; in this case, the profile as a whole is classed as conditionally unstable.

In the third case (right-hand panel) the atmosphere is saturated everywhere. Parcels near the surface are unstable even to infinitesimal perturbations, though parcels further up are stable; the profile as a whole is classed as unstable.

A word of warning: there is an alternative, and more traditional, definition of “conditional instability”, whereby a profile is conditionally unstable if its lapse rate is less than dry adiabatic but greater than moist adiabatic. This definition is fundamentally different from the finite-size perturbation definition given here: a profile that is conditionally unstable according to the traditional definition may actually be stable to all adiabatic parcel displacements of whatever size. Confusingly, the two definitions coexist and are sometimes mixed together. The conflict between the two definitions is a matter of current debate (see e.g. Sherwood, 2000). That such a basic definition should still be debated in a discipline over a century old is, among other things, a testament to the diversity of the those involved in meteorology,

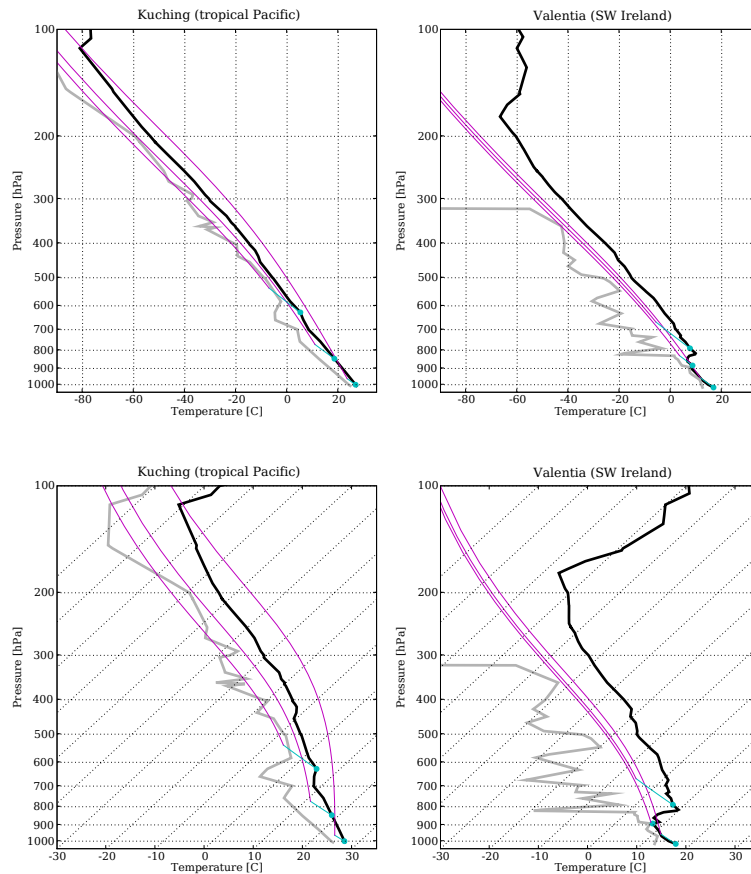


Figure 3.8: Real soundings from (left) Kuching, 6 Oct 2005 12Z, and (right) Valentia, 13 Oct 2006 12Z. Lower two panels show the same soundings plotted on a skew-T ln-p grid (the temperature isolines have been rotated clockwise by about 45°).

ranging from rough-and-ready practitioners to ivory-tower academic theorists. This diversity, and the ensuing communication problems, is both the bane and the charm of meteorology.

Now let's look at the stability of some real soundings. Fig. 3.8 shows two examples, one from Kuching in Malaysia and the other from Valentia in Ireland. In the Kuching sounding, surface parcels reach their LFC at around 900 hPa, and remain positively buoyant all the way up to about 120 hPa. Parcels higher up in the atmosphere are stable; overall, the sounding is conditionally unstable. The sounding at Valentia contains a strong *inversion* (a layer of the troposphere where temperature *increases* with height) which has a strongly stabilising

effect: parcels lifted from the surface experience some small positive buoyancy in a thin layer just under the inversion, but are negatively buoyant above the inversion. Overall, the profile may be classed as stable.

3.19 Skew-T and tephigram charts

Determining the stability of a sounding involves comparing the temperatures (and hence densities) of the sounding and of adiabatically lifted parcels. It is a fact of life that the troposphere is generally never *very* far from a moist adiabat—more precisely, the difference between a temperature sounding and a nearby moist adiabat is usually small compared with the overall temperature change from surface to tropopause. As a result, plots such as those along the top row in Fig. 3.8 are graphically inefficient: sounding and parcel trajectories are bunched up along the diagonal, with white space elsewhere. A neat trick to improve the presentation and make the important features stand out more clearly is to tilt the constant-temperature lines by 45° , as shown along the bottom row of Fig. 3.8: note how much more clearly (compared with Fig. 3.8) you can see the regions of positive and negative buoyancy, and how the shift from dry to moist adiabats is much more pronounced.

A *skew-T ln-p chart* is a special diagram used to plot atmospheric soundings. An example is shown in Fig. 3.9. Aside from pressure and tilted temperature lines, it has dry and moist adiabats at regular intervals, as well as lines showing dew-point temperature at fixed mixing ratio (so-called *mixing-ratio isopleths*; “isopleth” means “having the same value”). Since mixing ratio is conserved in an adiabatically-lifted unsaturated parcel, these lines permit quick identification of the LCL: given the temperature and dew-point of a parcel, just follow temperature up the dry adiabat and dew-point up the humidity isopleth until the two meet, and that’s the LCL.

A *tephigram chart* is very similar to a skew-T chart, but it uses as coordinates Θ and T ,

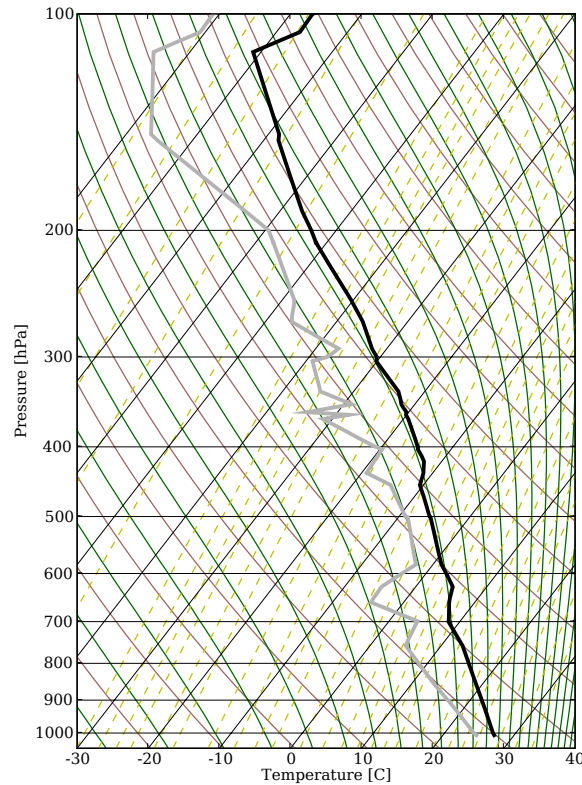


Figure 3.9: The Kuching sounding of Fig. 3.8 plotted on a skew-T ln-p chart. Black lines show pressure and temperature. Brown lines are dry adiabats, green lines moist adiabats, and yellow dashed lines show dew-point temperature at fixed mixing ratio.

with axes at right-angles¹. Since $\Theta = T(p/p_0)^{-Rd/c_{pd}}$, a line of constant p is a straight line making an angle $\arctan(p/p_0)^{-Rd/c_{pd}}$ to the horizontal. If we rotate this clockwise through 45° , then the $p = p_0 = 1000$ hPa line is horizontal, and lower-pressure lines are somewhat tilted. The result is very similar to a skew-T chart, but the dry adiabats are straight lines. An example is shown in Fig. 3.10.

¹Actually, the traditional choice of axis is T and $s = c_p \ln \Theta$, but since $\ln \Theta$ is essentially linear over the range of interest, it is simpler and almost equivalent to use Θ as an axis.

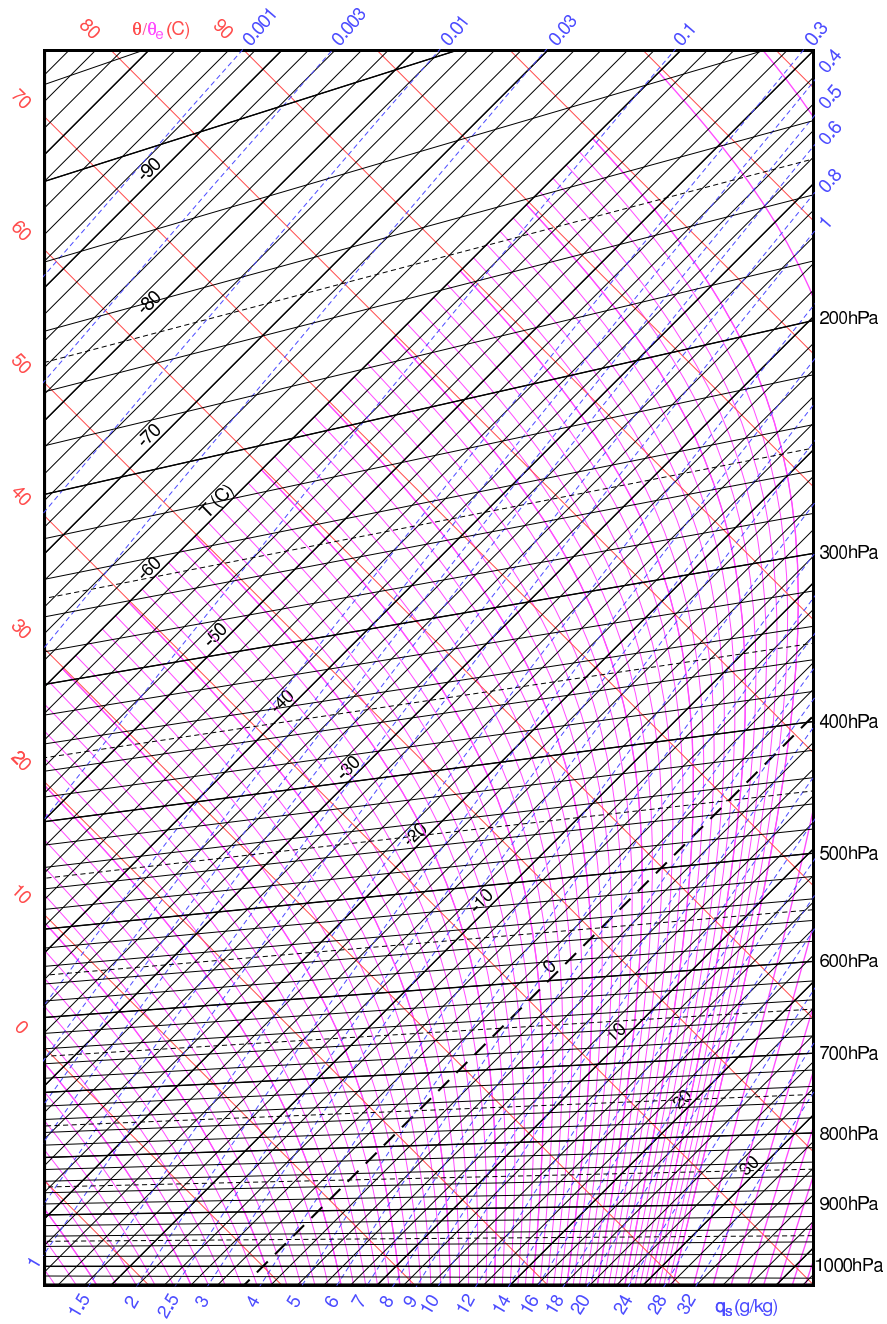


Figure 3.10: A tephigram chart, courtesy of Maarten Ambaum at the University of Reading. He has some interesting comments at <http://www.met.reading.ac.uk/sws97mha/Tephigram/>

3.20 CAPE and CINE

In Section 2.23, we defined CAPE as the work (per unit mass) done by the buoyancy force when a parcel ascends from its level of free convection (LFC) to its level of neutral buoyancy (LNB). This carries over to the moist case:

$$\text{CAPE} = R_d \int_{p_{\text{LNB}}}^{p_{\text{LFC}}} (T_{vp} - T_{vs}) d \ln p, \quad (3.78)$$

with the only difference that we use virtual temperature here. Graphically, this is the positive area between the parcel trajectory and sounding on a skew-T or tephigram plot. CAPE is positive for unstable and conditionally-unstable parcels, and zero for stable parcels. The actual value of CAPE is related to the intensity of the ensuing convection.

Between the surface and the LFC, the parcel is negatively buoyant, so work must be done to lift it to the LFC. This is called the *convective inhibition energy* (CINE), given by

$$\text{CINE} = R_d \int_{p_{\text{LFC}}}^{p_0} (T_{vp} - T_{vs}) d \ln p, \quad (3.79)$$

which is always negative. The greater the CINE, the more work is needed to lift parcels to their LFC and the more difficult it is to trigger convection. The presence of CINE means that CAPE is not immediately released as soon as it is generated, but can accumulate until an adequate triggering event occurs. Very large amounts of CAPE can then be released all at once, leading to very intense storms.

3.21 Stability indices and thunderstorm forecasting

Because CAPE is somewhat complicated to compute, a number of simpler “stability indices” have been devised over the years; these are numbers which, like CAPE, characterise the degree of instability of a profile, and are used in forecasting severe weather (such as thunderstorms). The most classic is the *Showalter index* (SI), defined by

$$SI = T_{500} - T'_{850} \quad (3.80)$$

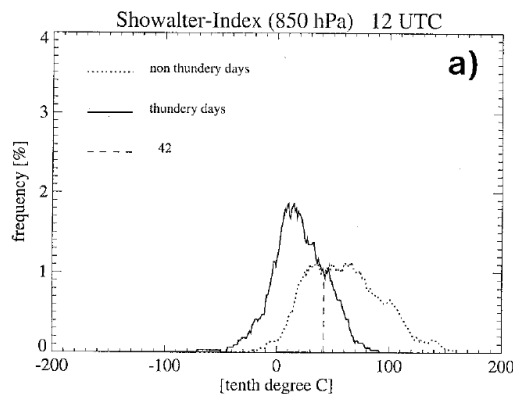


Figure 3.11: Probability distributions of the Showalter index conditional on a storm occurring (solid) and not occurring (dotted). From Huntrieser et al. (1997).

where T'_{850} is the temperature of a parcel lifted pseudoadiabatically from 850 to 500 hPa, and T_{500} is the temperature of the surroundings at 500 hPa. Clearly, positive SI indicates stable conditions. Another widely-used index is the *lifted index* (LI), defined by

$$LI = T_{500} - T'_s \quad (3.81)$$

where now T'_s is the temperature of a parcel lifted to 500 hPa from the surface. There are many other indices besides these two, all of which compare the temperature, humidity and winds at different levels.

Stability indices are indicators of convection and thunderstorms, but only in a statistical sense. Thus, given a profile with very negative SI (or LI), there is no guarantee that convection will occur, but there is a *greater probability* that it will. For stability indices to be quantitatively useful in forecasting, we need to know the *conditional probability* that convection will occur given that the index has a certain value (or, equivalently, the probability that the index has a certain value given that convection is observed). These probabilities can be estimated empirically, by computing the index for a great many soundings at a given site, and observing whether or not a thunderstorm develops. An example for SI in Switzerland is shown in Fig. 3.11. Once we have these probability distributions, we can make a probabilistic forecast: given a value of SI, we can say what is the probability that a thunderstorm will develop today. The catch is that the probability distributions are site-specific, so we cannot

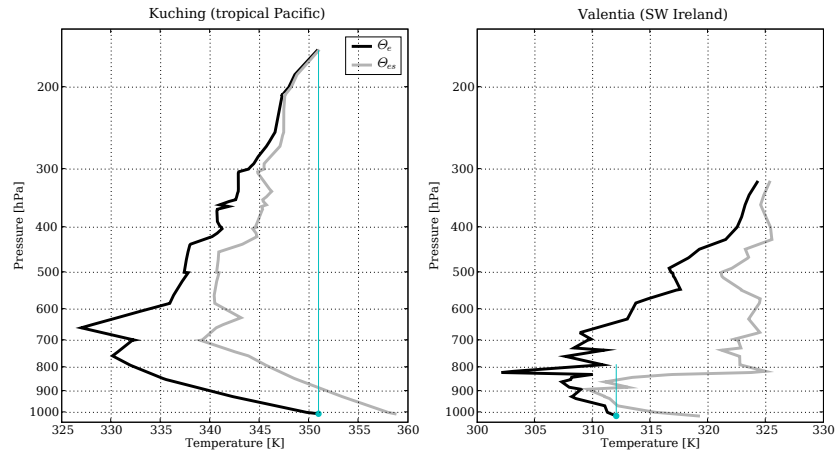


Figure 3.12: Θ_e and Θ_{es} for the soundings shown in Fig. 3.8. The blue line shows the (constant) Θ_e of a parcel lifted from the ground.

use data from one station to make predictions about faraway locations. In general, stability indices are used in a “fuzzy” way, as a qualitative indicator. Operational forecasters, after a few years of hands-on work, develop a seat-of-the-pants feel for how likely a thunderstorm is given SI and other data.

3.22 Relation between Θ_e , Θ_{es} and stability

We saw in Section 2.22 that the static stability of dry air is determined by the vertical rate of change of Θ : if Θ increases with height, the profile is stable, otherwise it is unstable. This is because *at a given pressure*, Θ depends only on T ; so if we bring two parcels adiabatically to the same pressure, the one with higher Θ will always be warmer and thus lighter than the other.

For a parcel that can undergo condensation, things get more complicated. The conserved quantity in this case is Θ_e , which depends on humidity as much as on temperature. Thus, two parcels at the same pressure and with the same Θ_e can have very different temperatures; there

is no simple relation between Θ_e and parcel buoyancies. However, we can get around this by comparing the Θ_e of the test parcel with the *saturated* Θ_e (i.e. Θ_{es}) of the environment. By making the environment saturated, we remove the dependency on humidity, so that the comparison reflects only differences in temperatures. It can be shown that *for a parcel that has achieved saturation* (i.e. one that is above its LCL) the approximate relation

$$\Theta' - \Theta \simeq \frac{\Theta'_e - \Theta_{es}}{1 + \beta} \quad (3.82)$$

is valid, where primes refer to the adiabatically lifted parcel and $\beta = (\ell_v/c_p)\partial w_s/\partial T$. Fig. 3.12 shows this for the two real soundings.

Exercise 3.22: Show that (3.82) is true. To do so, expand the exponential in Θ_e and Θ_{es} to first order in a Taylor series, take the difference, and use a further expansion on $w_s(T)$.

3.23 Mixing lines and contrails

To form condensation in air, i.e. to make a cloud, you need to either cool the air, make it moister, or both. The easiest, fastest and by far the most common way for air to cool sufficiently to form a cloud is by adiabatic expansion; hence the importance of lifting and the emphasis on adiabatic processes in all we've done up to here. But there are other ways to cool air, and though they play a less important role, they are worth discussing. These processes involve exchange of heat and/or mass between an air parcel and its surroundings, so they are *diabatic* (some prefer the term “non-adiabatic”; and why not a-adiabatic or a-nondiabatic?).

One example is *radiative cooling*. As we will see later, all bodies (including bodies of gas) spontaneously emit radiation, which carries away energy. Under the right conditions (at night and under clear skies with weak winds), the Earth's surface can cool dramatically through radiative loss (hence the cold desert nights we're all familiar with from Lawrence of Arabia films). The air in contact with the surface will cool by conduction, and if it is humid enough, a fog will form; this is called *radiation fog*.



Figure 3.13: Contrails.

Another way to cool air is by mixing it with colder air. This is what happens when you can see your breath on a cold day: the warm moist air coming out of your mouth mixes with colder ambient air and, if conditions are right, the resultant mix is supersaturated so condensation forms. A more dramatic example of the same phenomenon are *contrails* (short for *condensation trails*), the linear clouds stretching behind aeroplanes high in the sky (Fig. 3.13). Contrails are of interest to various people, among them the Air Force: it makes the generals look bad when their multi-zillion dollar stealth fighter has a large cloud pointing at it like a neon sign. A more PC interest in contrails derives from their possible role in enhancing global warming. If you've ever stopped to look at contrails for a while, you will have noticed that often they don't just disappear, but evolve into more horizontally-extensive cirrus clouds. These clouds are thin enough to let plenty of sunlight through, but they still trap infrared radiation quite strongly (again, more on this later), so they lead to a net warming.

To understand contrail formation, it helps to look at a figure like 3.14. A jet engine is a machine that takes air with ambient temperature and humidity (represented by point 4 in the figure), adds heat and water vapour to it (both resulting from combustion) and spews it back out again in a highly turbulent state (point 1). Because of the turbulence, the hot,

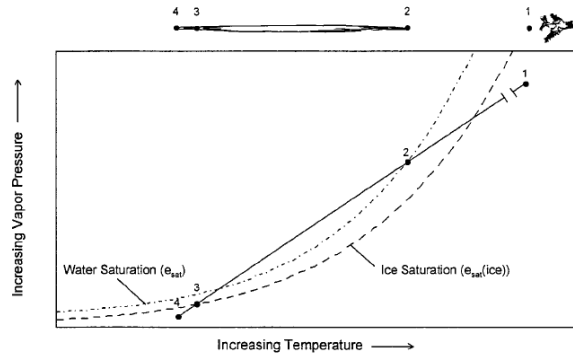


Figure 3.14: Schematic of contrail formation. From Schrader (1997).

moist exhaust air rapidly mixes with the much colder ambient air. If mass m_e of exhaust air mixes with mass m_a of ambient air, the temperature of the mixture is $T = (1 - f)T_e + fT_a$, where T_e is exhaust temperature, T_a is ambient temperature, and $f = m_e/(m_e + m_a)$ is the mixing fraction. A similar expression is valid for the humidity. As the exhaust air becomes more and more diluted, f increases and the point representing the state of the mixed air moves from point 1 to point 4 along a straight line called the *mixing line*. If the mixing line crosses the saturation vapour pressure curves, then a contrail will form. Normally, the ambient air is unsaturated, and so the contrail will eventually dissipate as the mixed air approaches point 4 (this is why contrails typically have a beginning and an end). However, it can happen that point 4 lies in between the ice and the water saturation curves (i.e., the air is supersaturated with respect to ice but unsaturated with respect to water). Under these conditions, clouds will not form spontaneously (because it is difficult to nucleate ice drops directly), but icy contrails will persist for a long time.

Exercise 3.23: The air coming out of a jet engine is much warmer than its surroundings and therefore very buoyant; you would expect it to shoot up into the sky like a balloon. However, this does not happen appreciably: contrails form roughly at the same level as the aeroplane. Give a quantitative (order-of-magnitude) explanation for this. It helps to look closely at Fig. 3.13 and make reasonable assumptions about the speed and length of the aeroplane and the temperature difference between exhaust and ambient air.